Problem Set

Problems borrowed from the Ein-Gedi Quantum Conference:
https://itaileigh.wixsite.com/quantum-kickoff/copy-of-posters
Solutions are also available there.

Exercise 1 (Distinguishing quantum states).

Let $|\psi_0\rangle$ and $|\psi_1\rangle$ be one qubit quantum states. If they have large overlap $|\langle\psi_0|\psi_1\rangle|$, then no quantum process can distinguish these two states with high probability. Let P be the process consisting of a measurement in basis $(|\phi_0\rangle, |\phi_1\rangle)$, then outputting the label x of the measured sate $|\phi_x\rangle$ as our guess. The effectiveness of state discrimination regarding $|\psi_0\rangle, |\psi_1\rangle$ can be quantified by

$$\Delta := |\Pr(P(|\psi_0\rangle) \text{ outputs } 0) - \Pr(P(|\psi_1\rangle) \text{ outputs } 0)|$$

Find a measurement for which

$$\Delta = \sqrt{1 - |\langle\psi_0|\psi_1\rangle|^2}.$$

(Hint: There is a 2-dimensional space containing $|\psi_0\rangle$ and $|\psi_1\rangle$. Think which orthonormal basis would distinguish these two states best. It may be useful to remember the trignometric identity $\cos 2x = 2\cos^2 x - 1$.)

In a future exercise we will show that this strategy is optimal.

Exercise 2 (Quantum states and measurements).

Consider the following *maximally entangled* state of 2 qubits

$$\frac{|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle}{\sqrt{2}}.$$

1. Prove that this state is entangled (i.e., it is not a tensor product state $|\alpha\rangle \otimes |\beta\rangle$ for any $|\alpha\rangle$ and $|\beta\rangle$.)
2. Suppose we measure the left qubit in some orthonormal basis $|\alpha\rangle, |\alpha^\perp\rangle$. What is the probability for the left qubit to be projected onto $|\alpha\rangle$? What is the state of the right qubit after the measurement (based on the outcome)?

Exercise 3 (Entanglement and inner product estimation).

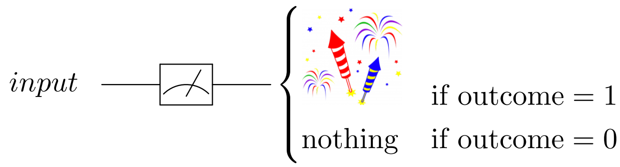
Consider the entangled state

$$\frac{|0\rangle \otimes |\alpha_0\rangle + |1\rangle \otimes |\alpha_1\rangle}{\sqrt{2}}$$

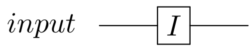
where the left register is a 1-qubit register, the right register is of arbitrary size, and the states $|\alpha_0\rangle$ and $|\alpha_1\rangle$ are normalized. What is the probability of getting the outcome $|+\rangle$ if we measure the left register in the $|+\rangle, |-\rangle$ basis (the eigenvectors of the X operator)? Express it as a function of $\langle\alpha_0|\alpha_1\rangle$.

Exercise 4 Elizur-Vaidman's fireworks.

Recall that we model a working quantum firework using the following circuit:



A dud is modeled as:



You are given either a dud or a working firework. Your goal is to successfully identify a working firework (without exploding it). In the lecture, we saw an algorithm which always identifies a dud as a dud, and it identifies a working firework as such with $\frac{1}{4}$ probability.

The goal of this exercise is to increase the success probability for the working firework. Fill in the details in the following algorithm:

1. Initialize the input qubit to the $|0\rangle$ state.
2. Set $\theta = \boxed{\ ?\ }$.
3. Repeat n times:
   (a) Apply $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ to the input qubit.
   (b) See if the potential firework lights off when using the qubit as an input. If it does, declare failure.
4. Measure the qubit in the standard basis. If the outcome is $\boxed{\ ?\ }$, output "definitely a dud", if the outcome is $\boxed{\ ?\ }$ output "definitely working".

Prove the following:

- A dud is always identified as a dud.
- A working firework is never identified as a dud, and that it explodes with probability at most $O(\frac{1}{n})$ (and therefore identified correctly with probability $1 - O(\frac{1}{n})$). Hint: Use the approximation $\sin(\theta) = \theta + O(\theta^2)$.